

Rainbow Colouring of Split Graphs

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Abstract

A *rainbow path* in an edge coloured graph is a path in which no two edges are coloured the same. A *rainbow colouring* of a connected graph G is a colouring of the edges of G such that every pair of vertices in G is connected by at least one rainbow path. The minimum number of colours required to rainbow colour G is called its *rainbow connection number*. Between them, Chakraborty et al. [J. Comb. Optim., 2011] and Ananth et al. [FSTTCS, 2012] have shown that for every integer k , $k \geq 2$, it is NP-complete to decide whether a given graph can be rainbow coloured using k colours.

A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. Chandran and Rajendraprasad have shown that the problem of deciding whether a given split graph G can be rainbow coloured using 3 colours is NP-complete and further have described a linear time algorithm to rainbow colour any split graph using at most one colour more than the optimum [COCOON, 2012]. In this article, we settle the computational complexity of the problem on split graphs and thereby discover an interesting dichotomy. Specifically, we show that the problem of deciding whether a given split graph can be rainbow coloured using k colours is NP-complete for $k \in \{2, 3\}$, but can be solved in polynomial time for all other values of k .

Keywords: rainbow connectivity, rainbow colouring, split graphs, complexity.

1 Introduction

An *edge colouring* of a graph is a function from its edge set to the set of natural numbers. A path in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path*. An edge coloured graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a colouring is called a *rainbow colouring* of the graph. A rainbow colouring using minimum possible number of colours is called *optimal*. The minimum number of colours required to rainbow colour a connected graph G is called its *rainbow connection*

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number, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, that of an even cycle is half its length, and that of a tree is its number of edges. Note that disconnected graphs cannot be rainbow coloured and hence their rainbow connection number is left undefined. Any connected graph can be rainbow coloured by giving distinct colours to the edges of a spanning tree of the graph. Hence the rainbow connection number of any connected graph is less than its number of vertices. It is trivial to see that that $rc(G)$ is at least the diameter of G . It is easy to see that no two bridges in a graph can get the same colour under a rainbow colouring and hence $rc(G)$ is lower bounded by the number of bridges in the G .

The concept of rainbow colouring was introduced by Chartrand, Johns, McKeon, and Zhang in [7] where they also determined the precise values of rainbow connection number for some special graphs. Subsequently, there have been various investigations towards finding good upper bounds for rainbow connection number in terms of other graph parameters [4, 14, 16, 3] and for many special graph classes [12, 16, 2]. Behaviour of rainbow connection number in random graphs is also well studied [4, 10, 15, 9]. A basic introduction to the topic can be found in Chapter 11 of the book *Chromatic Graph Theory* by Chartrand and Zhang [8] and a survey of most of the recent results in the area can be found in the article by Li and Sun [11] and also in their monograph *Rainbow Connection of Graphs* [13].

2 Our contribution

In this article we focus on the computational complexity of the following decision problem on split graphs (Definition 5).

Problem 1 ($\text{RAINBOWCOLOUR}(G, k)$). Given a connected graph G and a positive integer k , decide whether G can be rainbow coloured using k colours.

The first result showing the computational complexity of the above problem was due to Chakraborty, Fischer, Matsliah, and Yuster [5]. They showed that it is **NP**-hard to compute the rainbow connection number of an arbitrary graph. In particular, it was shown that the problem $\text{RAINBOWCOLOUR}(G, 2)$ is **NP**-complete. Later, Ananth, Nasre, and Sarpatwar [1] complemented the above result and now we know that for every integer k , $k \geq 2$, the problem $\text{RAINBOWCOLOUR}(G, k)$ is **NP**-complete. This prompts one to look at the computational complexity of the problem on special graph classes. Chandran and Rajendraprasad have shown that $\text{RAINBOWCOLOUR}(G, k)$ is solvable in linear time for threshold graphs, **NP**-complete on split graphs for $k = 3$ and **NP**-complete on chordal graphs for all $k \geq 3$ [6]. It is easy to see that complete graphs alone can be rainbow coloured using 1 colour. The complexity of the problem $\text{RAINBOWCOLOUR}(G, k)$ on chordal graphs for $k = 2$ and split graphs for all positive integers k except 1 and 3 was left open. In this article, we solve the same and thereby discover the following interesting dichotomy.

Theorem 1. *The problem $\text{RAINBOWCOLOUR}(G, k)$ on split graphs is **NP**-complete for $k \in \{2, 3\}$ and polynomial-time solvable for all other values of k .*

3 On the proofs

First we show that the problem $\text{RAINBOWCOLOUR}(G, k)$ is polynomial time solvable for $k \geq 4$ by demonstrating the following structural result whose proof is given in Appendix A.2. Let $\text{pen}(G)$ denotes the set of pendant vertices (vertices with exactly one neighbour) in a graph G .

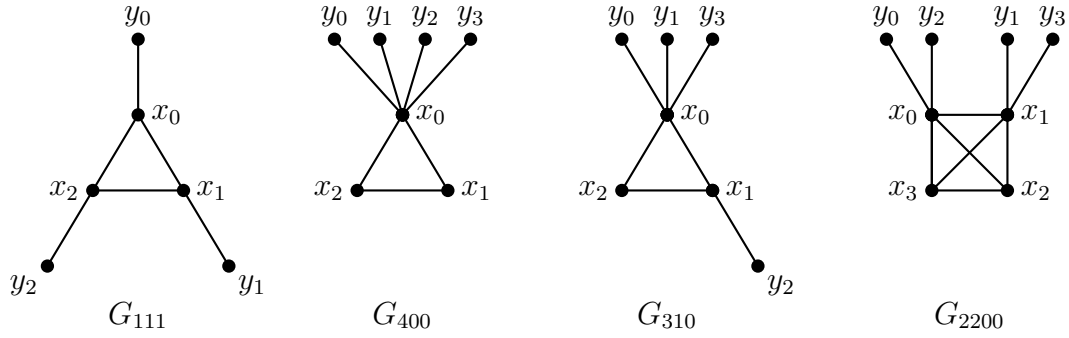


Figure 1: Four special split graphs which constitute the set \mathcal{G}

Lemma 2. *If a split graph G , under some isomorphism, contains any of the graphs $H \in \mathcal{G}$ in Figure 1 as a subgraph with $\text{pen}(H) \subseteq \text{pen}(G)$, then $rc(G) = |\text{pen}(G)|$.*

From the above lemma and the easy observation that $rc(G) \geq |\text{pen}(G)|$ for any graph, it follows that for each integer $k \geq 4$ there exists a polynomial time algorithm to check if the rainbow connection number of a split graph is at most k . The proof gives an explicit rainbow colouring of G using $|\text{pen}(G)|$ colours if it contains any of the graphs $H \in \mathcal{G}$ as a subgraph with $\text{pen}(H) \subset \text{pen}(G)$, and thus we show that any split graph with rainbow connection number at least 4 can be optimally rainbow coloured in polynomial time (Corollary 6 in Appendix A.2).

Next we show that the problem $\text{RAINBOWCOLOUR}(G, 2)$ remains NP-complete for split graphs. This is established by showing a two-step reduction. Given a graph $G = (V, E)$, and a collection of subsets \mathcal{S} of V , the problem $\text{BICLIQUECOVER}(G, \mathcal{S})$ is to decide whether there exists a *bipartitioning function* $X : \mathcal{S} \rightarrow 2^V$ such that $X(T) \subset T, \forall T \in \mathcal{S}$ and G is covered by the family of bicliques $\{(X(T), T \setminus X(T)) : T \in \mathcal{S}\}$. We show that 3-SAT is reducible to BICLIQUECOVER which in turn is reducible to $\text{RAINBOWCOLOUR}(G, 2)$ with G being a split graph (Lemmata 7 and 8 in Appendix A.3).

4 Consequences

The problems below are only superficially different from the $\text{RAINBOWCOLOUR}(G, 2)$ problem on split graphs (see the discussion after Problem 6 in Appendix A.3) and hence we deduce that they are also NP-complete (the problem size being $O(mn)$ in each case).

Problem 2 ($\text{ENSUREDISTINCTROWS}(C)$). Given a subset $C \subset [m] \times [n]$ of locations, decide whether there exists an $m \times n$ matrix M with entries from $\{0, 1\}$ such that any two rows of M will remain distinct, no matter what changes are made to the entries of M at locations in C .

Problem 3 ($\text{ORTHOGONALPACKING}(B)$). Given a set B of m n -dimensional boxes whose sides are either 1 or $1/2$ in each dimension, decide whether they can be packed without rotation into an n -dimensional unit cube.

We would also like to emphasise that the problem $\text{RAINBOWCOLOUR}(G, 2)$ is known to be linear time solvable for threshold graphs, which are split graphs in which the neighbourhoods of the independent set vertices form a total order under inclusion. In particular a threshold graph G can be rainbow coloured using 2 colours if and only if the degrees of the vertices in a maximum independent set I of G satisfy the Kraft's inequality, viz. $\sum_{v \in I} 2^{-d(v)} \leq 1$ where $d(v)$ denotes the degree of a vertex v [6]. The problem $\text{ENSUREDISTINCTROWS}(C)$ can be viewed as a combinatorial generalisation of the problem of constructing a prefix-free code given a set of desired lengths. The latter is poly-time solvable while the above generalisation is shown here to be NP-complete.

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A Appendix

A.1 Notation and definitions

All graphs considered in this article are finite, simple and undirected. For a graph G , we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively. An edge $\{u, v\}$ in a graph may be denoted by uv to reduce clutter. Unless mentioned otherwise, n and m will respectively denote the number of vertices and edges of the graph in consideration. The subgraph of G induced on a vertex set $S \subset V(G)$ is denoted by $G[S]$.

The shorthand $[n]$ denotes the set $\{1, \dots, n\}$. The cardinality of a set S is denoted by $|S|$ and the family of all subsets of S is denoted by 2^S . The union of two disjoint sets A and B is denoted by $A \dot{\cup} B$.

Definition 3. Let G be a connected graph. The *length* of a path is its number of edges. The *distance* between two vertices u and v in G , denoted by $d(u, v)$ is the length of a shortest path between them in G . The *diameter* of G is $\text{diam}(G) := \max_{u, v \in V(G)} d(u, v)$.

Definition 4. The *neighbourhood* $N(v)$ of a vertex v is the set of vertices adjacent to v but not including v . A vertex is called *pendant* if its degree is 1. An edge incident on a pendant vertex is called a *pendant edge* and the set of pendant vertices of a graph G is denoted by $\text{pen}(G)$.

Definition 5. A graph G is called *chordal*, if there is no induced cycle of length greater than 3. A graph G is a *split graph*, if $V(G)$ can be partitioned into a clique and an independent set. A graph G is a *threshold graph*, if there exists a weight function $w : V(G) \rightarrow \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.

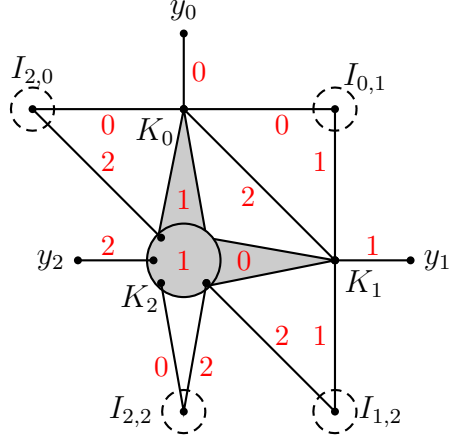
A.2 More than three colours: Polynomial time solution

Proof of Lemma 2

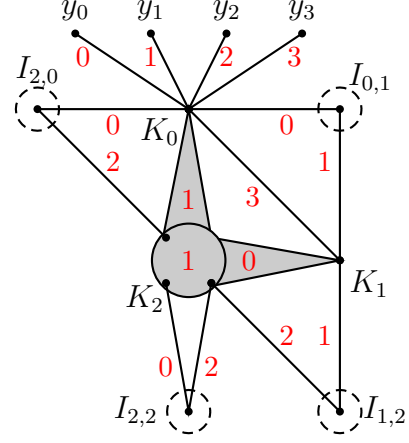
Statement. If a split graph G , under some isomorphism, contains any of the graphs $H \in \mathcal{G}$ in Figure 1 as a subgraph with $\text{pen}(H) \subseteq \text{pen}(G)$, then $rc(G) = |\text{pen}(G)|$.

Let us relabel the vertices of G so that H is contained as a (labelled) subgraph of G with $\text{pen}(H) \subset \text{pen}(G)$. First we note that it suffices to prove the statement when $\text{pen}(G) = \text{pen}(H)$. Suppose $P' = \text{pen}(G) \setminus \text{pen}(H)$ is non-empty. Then consider the induced subgraph G' of G obtained by removing all the vertices in P' . Note that G' also has H as a subgraph with $\text{pen}(H) \subset \text{pen}(G')$. If G' can be rainbow coloured with $|\text{pen}(G')|$ colours, we can easily extend this to a rainbow colouring of G with p colours by giving a new colour to each edge of G incident to a vertex in P' . Henceforth in this proof we assume $\text{pen}(G) = \text{pen}(H)$.

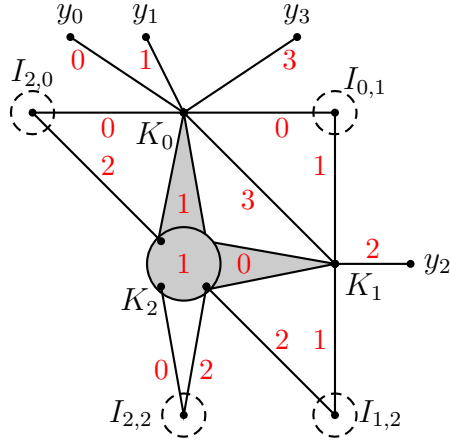
The proof is divided into four cases based on $H \in \mathcal{G}$. In each case, we describe an edge-colouring c_G of G using $|\text{pen}(H)|$ colours and then show that c_G makes G rainbow connected. A partial illustration of the colourings is given in Figure 2. In each case, we set K to be a maximal clique in G , $I = V(G) \setminus K$, $P = \text{pen}(G)$ and $I' = I \setminus P$. For each $v \in I'$, we can assume that v has exactly 2 neighbours in K . Remaining edges from I' to K are not used in our colouring and hence may be assumed absent. In the first three cases below, that is when $H \in \{G_{111}, G_{400}, G_{310}\}$, we partition K and I' as follows. Vertices in K are grouped into 3 parts $K_0 = \{x_0\}$, $K_1 = \{x_1\}$ and $K_2 = K \setminus \{x_0, x_1\}$ while the vertices in I' are grouped into 4 parts $I_{0,1}$, $I_{1,2}$, $I_{2,0}$ and $I_{2,2}$, where $I_{i,j}$, $i \neq j$, consists of those vertices in I' with one neighbour in K_i and one neighbour in K_j and $I_{2,2}$ consists of those vertices in I' with both neighbours in K_2 . In the fourth case, K is partitioned into 4 parts $K_i = \{x_i\}$, $i \in \{0, 1, 2\}$, and $K_3 = K \setminus \{x_0, x_1, x_2\}$ while I' is partitioned into 7 parts $I_{i,j}$, $\{i, j\} \subset \mathbb{Z}_4$, $i \neq j$, and $I_{3,3}$ as before. While defining a colouring c_G of $E(G)$, we will use the shorthand $c_G(A, B) = i$ to indicate that $c_G(\{a, b\}) = i$, for all $\{a, b\} \in E(G)$ such that $a \in A$ and $b \in B$.



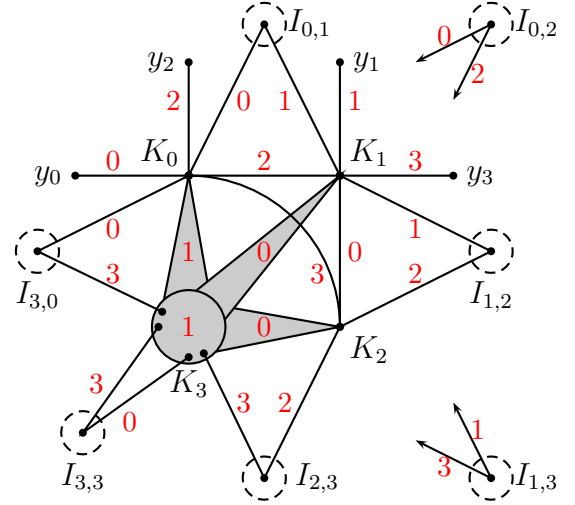
Case 1 (Equation 1)



Case 2 (Equation 2)



Case 3 (Equation 3)



Case 4 (Equation 4)

Figure 2: A partial illustration of the four colourings defined in the proof of Lemma 2. Only one representative vertex from the each part of the independent set is illustrated. The edge-colours are indicated in red to distinguish them from other labels.

Case 1 ($H = G_{111}$).

In this case, $\text{pen}(G) = \{y_0, y_1, y_2\}$ and thus $p = 3$. We define the 3-colouring $c_G : E(G) \rightarrow \mathbb{Z}_3$ (See Figure 2).

$$\begin{aligned} c_G(K_i, K_j) &= k, \text{ where } \{i, j, k\} = \mathbb{Z}_3, \\ c_G(K_2, K_2) &= 1, \\ c_G(K_i, I \setminus I_{2,2}) &= i, \quad \forall i \in \mathbb{Z}_3, \text{ and} \\ c_G(\{v, u_l\}) &= l, \quad \forall v \in I_{2,2}, l \in \{0, 2\} \text{ and } N(v) = \{u_0, u_2\}. \end{aligned} \tag{1}$$

Now we show that c_G is a rainbow colouring of G by listing down a rainbow path between every pair of vertices which are at a distance of at least 2 from each other. Let I_i , $i \in \{0, 1, 2\}$, denote the set of vertices in I with at least one neighbour in K_i . Note that a vertex in $I_{i,j}$ is part of I_i and I_j and hence two distinct vertices in $I_{i,j}$ are connected by a rainbow path of the second type in the list below.

$$\begin{aligned} u \in I_i \text{ to } v \in K_j, \ i \neq j : u &\xrightarrow{i} K_i \xrightarrow{k} K_j & (\text{where } \{i, j, k\} = \mathbb{Z}_3) \\ u \in I_i \text{ to } v \in I_j, \ i \neq j : u &\xrightarrow{i} K_i \xrightarrow{k} K_j \xrightarrow{j} v & (\text{where } \{i, j, k\} = \mathbb{Z}_3) \\ u \in I_{2,2} \cup \{y_2\} \text{ to } v \in K_2, \ v \notin N(u) : u &\xrightarrow{2} K_2 \xrightarrow{1} v \\ u \in I_{2,2} \cup \{y_2\} \text{ to } v \in I_{2,2}, \ v \neq u : u &\xrightarrow{2} K_2 \xrightarrow{1} K_2 \xrightarrow{0} v \end{aligned}$$

Case 2 ($H = G_{400}$).

In this and next two cases, $\text{pen}(G) = \{y_0, \dots, y_3\}$ and thus $p = 4$. We define a 4-colouring $c_G : E(G) \rightarrow \mathbb{Z}_4$.

$$\begin{aligned} c_G(K_0, K_1) &= 3, \\ c_G(K_1, K_2) &= 0, \\ c_G(K_2, K_0) &= 1, \\ c_G(K_2, K_2) &= 1, \\ c_G(\{y_i, x_0\}) &= i, \quad \forall i \in \mathbb{Z}_4, \\ c_G(K_i, I' \setminus I_{2,2}) &= i, \quad \forall i \in \{0, 1, 2\}, \text{ and} \\ c_G(\{v, u_l\}) &= l, \quad \forall v \in I_{2,2}, l \in \{0, 2\} \text{ and } N(v) = \{u_0, u_2\}. \end{aligned} \tag{2}$$

Notice that the colouring defined by Equation 2 is similar to that defined by Equation 1 except for the pendant edges and the clique edge $\{x_0, x_1\}$. Now we show that c_G is a rainbow colouring of G by listing down a rainbow path between every pair of vertices which are at a distance of at least 2 from each other. This time, let I_i , $i \in \{0, 1, 2\}$, denote the set of vertices in $I \setminus \{y_1, y_2, y_3\}$ with at least one neighbour in K_i .

$$\begin{aligned} I_i \text{ to } K_j \text{ then } I_j, \ i \neq j : I_i &\xrightarrow{i} K_i \xrightarrow{k} K_j \xrightarrow{j} I_j \text{ (where } k \in \mathbb{Z}_4 \setminus \{i, j\}) \\ I_{2,2} \text{ to } K_2 \text{ then } I_{2,2} : I_{2,2} &\xrightarrow{2} K_2 \xrightarrow{1} K_2 \xrightarrow{0} I_{2,2} \\ \{y_1, y_2, y_3\} \text{ to } v \in I_{2,0} \cup I_{0,1} : y_i &\xrightarrow{i} K_0 \xrightarrow{0} v \\ y_1 \text{ to } K_1 \text{ then } K_2 \text{ then } I_{2,2} \cup I_{1,2} : y_1 &\xrightarrow{1} K_0 \xrightarrow{3} K_1 \xrightarrow{0} K_2 \xrightarrow{2} I_{2,2} \cup I_{1,2} \\ y_2 \text{ to } K_2 \text{ then } I_{2,2} : y_2 &\xrightarrow{2} K_0 \xrightarrow{1} K_2 \xrightarrow{0} I_{2,2} \\ y_2 \text{ to } K_1 \text{ then } I_{1,2} : y_2 &\xrightarrow{2} K_0 \xrightarrow{3} K_1 \xrightarrow{1} I_{1,2} \\ y_3 \text{ to } K_2 \text{ then } I_{2,2} \cup I_{1,2} : y_3 &\xrightarrow{3} K_0 \xrightarrow{1} K_2 \xrightarrow{2} I_{2,2} \cup I_{1,2} \end{aligned}$$

$$y_3 \text{ to } K_1 : y_3 \xrightarrow{3} K_0 \xrightarrow{1} K_2 \xrightarrow{0} K_1$$

Case 3 ($H = G_{310}$).

The colouring $c_G : E(G) \rightarrow \mathbb{Z}_4$ that we define in this case is similar to Case 2. The only difference is that the pendant vertex y_2 is now adjacent to x_1 instead of x_0 .

$$\begin{aligned} c_G(K_0, K_1) &= 3, \\ c_G(K_1, K_2) &= 0, \\ c_G(K_2, K_0) &= 1, \\ c_G(K_2, K_2) &= 1, \\ c_G(\{y_i, x_0\}) &= i, \quad \forall i \in \{0, 1, 3\}, \\ c_G(\{y_2, x_1\}) &= 2, \\ c_G(K_i, I' \setminus I_{2,2}) &= i, \quad \forall i \in \{0, 1, 2\}, \text{ and} \\ c_G(\{v, u_l\}) &= l, \quad \forall v \in I_{2,2}, l \in \{0, 2\} \text{ and } N(v) = \{u_0, u_2\}. \end{aligned} \tag{3}$$

Since all pairs of vertices not involving y_2 are connected by rainbow paths as described in Case 2, we only indicate below rainbow paths from y_2 to every other vertex in G to claim that c_G rainbow connects G .

$$\begin{aligned} y_2 \text{ to } v \in I_{0,1} \cup I_{1,2} : y_2 &\xrightarrow{2} K_1 \xrightarrow{1} v \\ y_2 \text{ to } K_0 \text{ then } K_2 \text{ then } I_{2,2} : y_2 &\xrightarrow{2} K_1 \xrightarrow{3} K_0 \xrightarrow{1} K_2 \xrightarrow{0} I_{2,2} \\ y_2 \text{ to } v \in \{y_0\} \cup I_{2,0} : y_2 &\xrightarrow{2} K_1 \xrightarrow{3} K_0 \xrightarrow{0} v \\ y_2 \text{ to } y_1 : y_2 &\xrightarrow{2} K_1 \xrightarrow{3} K_0 \xrightarrow{1} y_1 \\ y_2 \text{ to } y_3 : y_2 &\xrightarrow{2} K_1 \xrightarrow{0} K_2 \xrightarrow{1} K_0 \xrightarrow{3} y_3 \end{aligned}$$

Case 4 ($H = G_{2200}$).

Recall that in this case, unlike the previous three cases, we have partitioned K into 4 parts and I' into 7 parts. The colouring $c_G : E(G) \rightarrow \mathbb{Z}_4$ is defined as follows (See Figure 2).

$$\begin{aligned} c_G(K_i, K_{i+1}) &= i + 2, \quad i \in \{0, 2, 3\}, \\ c_G(K_1, K_2) &= 0, \\ c_G(K_i, K_{i+2}) &= i + 3, \quad i \in \{0, 1\}, \\ c_G(K_3, K_3) &= 1, \\ c_G(K_i, I' \setminus I_{3,3}) &= i, \quad \forall i \in \mathbb{Z}_4, \\ c_G(\{v, u_l\}) &= l, \quad \forall v \in I_{3,3}, l \in \{0, 3\} \text{ and } N(v) = \{u_0, u_3\}, \\ c_G(\{y_i, x_0\}) &= i \in \{0, 2\}, \text{ and} \\ c_G(\{y_i, x_1\}) &= i \in \{1, 3\}. \end{aligned} \tag{4}$$

Now we show that c_G is a rainbow colouring of G by listing down a rainbow path between every pair of vertices which are at a distance of at least 2 from each other. This time, let I_i , $i \in \mathbb{Z}_4$, denote the set of vertices in $I \setminus \{y_2, y_3\}$ with at least one neighbour in K_i . Notice that, as in the previous cases, the edge(s) between every K_i and K_j , $i \neq j$, is given a colour different from i and j . This ensures rainbow paths between I_i and $K_j \cup I_j$, and we need to work hard only to identify rainbow paths from y_2 and y_3 to rest of the graph.

$$\begin{aligned} I_i \text{ to } K_j \text{ then } I_j, \ i \neq j : I_i &\xrightarrow{i} K_i \xrightarrow{k} K_j \xrightarrow{j} I_j \text{ (where } k \in \mathbb{Z}_4 \setminus \{i, j\}) \\ I_{3,3} \text{ to } K_3 \text{ then } I_{3,3} : I_{3,3} &\xrightarrow{3} K_3 \xrightarrow{1} K_3 \xrightarrow{0} I_{3,3} \end{aligned}$$

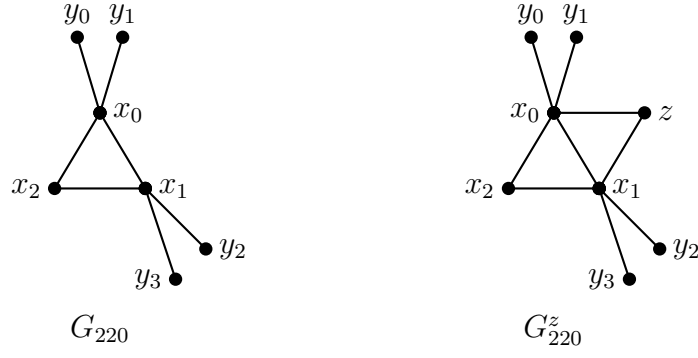


Figure 3: The graphs G_{220} and G_{220}^z mentioned in the proof of Corollary 6.

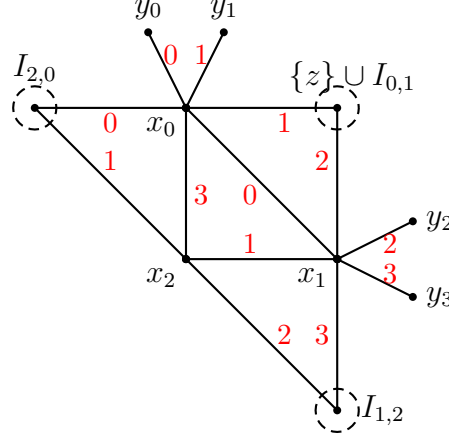


Figure 4: A partial illustration of the colouring of G when it contains G_{220}^z as a subgraph. Part of proof of Corollary 6

$$\begin{aligned}
& y_2 \text{ to } I_0 : y_2 \xrightarrow{2} K_0 \xrightarrow{0} I_0 \\
& y_2 \text{ to } K_2 \text{ then } K_1 \text{ then } I_1 : y_2 \xrightarrow{2} K_0 \xrightarrow{3} K_2 \xrightarrow{0} K_1 \xrightarrow{1} I_1 \\
& y_2 \text{ to } K_3 \text{ then } I_3 : y_2 \xrightarrow{2} K_0 \xrightarrow{1} K_3 \xrightarrow{3} I_3 \\
& y_2 \text{ to } y_3 : y_2 \xrightarrow{2} K_0 \xrightarrow{1} K_3 \xrightarrow{0} K_1 \xrightarrow{3} y_3 \\
& y_3 \text{ to } I_1 : y_3 \xrightarrow{3} K_1 \xrightarrow{1} I_1 \\
& y_3 \text{ to } K_0 \text{ then } I_0 : y_3 \xrightarrow{3} K_1 \xrightarrow{2} K_0 \xrightarrow{0} I_0 \\
& y_3 \text{ to } K_2 \text{ then } I_2 : y_3 \xrightarrow{3} K_1 \xrightarrow{0} K_2 \xrightarrow{2} I_2 \\
& y_3 \text{ to } K_3 \text{ then } I_{3,3} : y_3 \xrightarrow{3} K_1 \xrightarrow{2} K_0 \xrightarrow{1} K_3 \xrightarrow{0} I_{3,3}
\end{aligned}$$

Though we haven't indicated rainbow paths from y_2 to I_2 and y_3 to I_3 , since $I_2 \subset I_0 \cup I_1 \cup I_3$ and $I_3 \subset I_0 \cup I_1 \cup I_2 \cup I_{3,3}$, we have exhausted all pairs of vertices in the list above.

□

Corollary 6. *For each integer $k \geq 4$ there exists a polynomial time algorithm to check if the rainbow connection number of a split graph is at most k . Furthermore, any split graph with rainbow connection number at least 4 can be optimally rainbow coloured in polynomial time.*

Proof. Let G be a split graph with p pendant vertices. If G is a tree, then $rc(G)$ is equal to the number of edges in G and so we can check in linear time if $rc(G) \leq k$ for any k . In the case when G is not a tree, a maximal clique K in G contains at least 3 vertices. Fix any $k \geq 4$. If $p \leq k - 1$, we know from [6, Corollary 2] that $rc(G) \leq k$. Similarly if $p > k$, then $rc(G) > k$. Hence we can assume that $p = k$. Let K' be the vertices in K which are adjacent to at least one pendant vertex of G . If $|K'| \geq 3$, then G contains G_{111} as a subgraph with $\text{pen}(G_{111}) \subset \text{pen}(G)$. If $|K'| \leq 2$ and $G[K \cup \text{pen}(G)]$ is not isomorphic to G_{220} (Figure 3), then G contains $H \in \{G_{400}, G_{310}, G_{2200}\}$ as a subgraph with $\text{pen}(H) \subseteq \text{pen}(G)$. In all the cases above, it follows from Lemma 2 that $rc(G) \leq k$ and the proof therein gives a rainbow colouring in polynomial time.

If $G[K \cup \text{pen}(G)]$ is isomorphic to G_{220} then let us relabel $V(G)$ so that G_{220} is a subgraph of G . It is not difficult to see that if G has G_{220}^z as a subgraph for some $z \in V(G)$ then the $rc(G) = 4$. See Figure 4 for a partial illustration of one possible rainbow colouring. Conversely, in any attempted rainbow colouring of G using 4 colours, the 4 pendant edges $x_i y_j, i \in \{0, 1\}, j \in \{0, \dots, 3\}$ have to get 4 different colours and the edge $x_0 x_1$ has to reuse one of these 4 colours, say the one used by $x_0 y_0$ (as it is in Figure 4). Then it is easy to see that we need at least two more 2-length paths between x_0 and x_1 so as to provide rainbow paths from y_0 to y_2 and y_3 , which is available only if G has a subgraph isomorphic to G_{220}^z . \square

A.3 Two colours: NP-completeness

In order to show that $\text{RAINBOWCOLOUR}(G, 2)$ is NP-complete, we will also use the following two decision problems:

Problem 4 ($\text{BICLIQUECOVER}(G, \mathcal{S})$). Given a graph $G = (V, E)$, and a collection of subsets \mathcal{S} of V , decide whether there exists a *bipartitioning function* $X : \mathcal{S} \rightarrow 2^V$ such that $X(T) \subset T, \forall T \in \mathcal{S}$ and for every edge $\{u, v\} \in E(G)$ there exists a $T \in \mathcal{S}$ with $u \in X(T)$ and $v \in T \setminus X(T)$.

Problem 5 ($3\text{-SAT}(\varphi)$). Given a boolean formula φ in which every clause contains exactly 3 distinct literals corresponding to three distinct variables, decide whether there exists an evaluation of variables of φ such that every clause contains at least one satisfied literal.

Next two lemmata show a reduction of $3\text{-SAT}(\varphi)$ to $\text{BICLIQUECOVER}(G, \mathcal{S})$ (Lemma 8) and a reduction of $\text{BICLIQUECOVER}(G, \mathcal{S})$ to $\text{RAINBOWCOLOUR}(G, 2)$ on split graphs (Lemma 7). Problem 5 is known to be NP-complete since general SAT can be easily reduced to our version of 3-SAT problem. Note that all the three problems clearly belong to class NP. It means that polynomial time reductions from $3\text{-SAT}(\varphi)$ to $\text{BICLIQUECOVER}(G, \mathcal{S})$ and $\text{RAINBOWCOLOUR}(G, 2)$ is enough to show NP-completeness of the latter. It can be easily seen that the reductions used in proofs of Lemmata 7 and 8 are polynomial. Thus we show that $\text{RAINBOWCOLOUR}(G, 2)$ is NP-complete on split graphs (Theorem 9).

Lemma 7. *Problem $\text{BICLIQUECOVER}(G, \mathcal{S})$ is reducible to $\text{RAINBOWCOLOUR}(G', 2)$ where G' is a split graph.*

Proof. Let (G, \mathcal{S}) be an instance of BICLIQUECOVER and $G = (V, E)$. Let $\bar{E} = \binom{V}{2} \setminus E$ be the set of edges of complement of G . We define split graph $G' = (A' \dot{\cup} B', E')$ in the following way (see Figure 5):

$$A' = \bigcup_{v \in V} \{u'_v\} \cup \bigcup_{T \in \mathcal{S}} \{s_T\} \cup \bigcup_{e \in \bar{E}} \{x_e\}$$

$$B' = \bigcup_{v \in V} \{u_v\}$$

$$E' = \binom{A'}{2} \cup \bigcup_{v \in V} \{u_v u'_v\} \cup \bigcup_{v \in T \in \mathcal{S}} \{u_v s_T\} \cup \bigcup_{e \in \bar{E}; e=vw} \{u_v x_e, u_w x_e\}$$

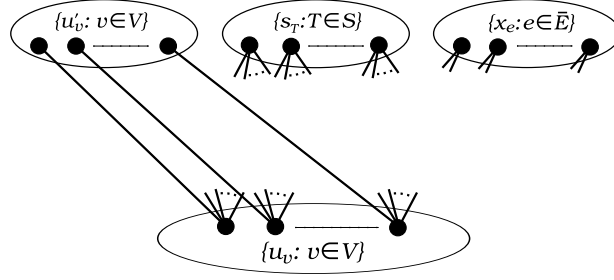


Figure 5: Graph $G' = (A' \dot{\cup} B', E')$.

We prove that $rc(G') \leq 2$ if and only if (G, \mathcal{S}) is a “yes” instance of $\text{BICLIQUECOVER}(G, \mathcal{S})$. At first suppose that (G, \mathcal{S}) is a “yes” instance of BICLIQUECOVER . Let $X: \mathcal{S} \rightarrow 2^V$ be a function such that bi-cliques $\{(X(T), T \setminus X(T)) : T \in \mathcal{S}\}$ cover all edges of G . We define coloring $col: E' \rightarrow \{red, blue\}$ of edges of G' in the following way:

- $col(e') = blue$, if $e' \in \binom{A'}{2}$
- $col(e') = red$, if $e' = u_v u'_v$ and $v \in V$
- $col(e') = blue$, if $e' = u_v s_T$, $T \in \mathcal{S}$ and $v \in X(T)$
- $col(e') = red$, if $e' = u_v s_T$, $T \in \mathcal{S}$ and $v \in T \setminus X(T)$
- For every $e = vw \in \bar{E}$ we set $col(u_v x_e)$ and $col(u_w x_e)$ in such a way that $col(u_v x_e) \neq col(u_w x_e)$.

We will show that for this coloring there exists a rainbow path between any two vertices of G' . If $u, u' \in A'$ then $uu' \in E'$ and we are done. If $u_v \in B'$ and $u' \in A'$ then either $u' = u'_v$ and $u_v u'_v \in E'$ or we can take rainbow path u_v, u'_v, u' . If $u_v, u_w \in B'$ then either $e = vw \in \bar{E}$ and the path u_v, x_e, u_w is rainbow or $vw \in E$, in which case there exists $T \in \mathcal{S}$ such that bi-clique $(X(T), T \setminus X(T))$ covers vw and the path u_v, s_T, u_w is rainbow.

For the opposite direction suppose that $col: E' \rightarrow \{red, blue\}$ is a coloring of edges of G' such that there exists a rainbow path between any two vertices of G' . We define mapping $X: \mathcal{S} \rightarrow 2^V$ in such a way that $v \in X(T)$ if and only if $v \in T \in \mathcal{S}$ and $col(u_v, s_T) = blue$.

Suppose that vw is an edge of G . From the definition of G' we know that all paths from u_v to u_w in G of length at most 2 are of the form u_v, s_T, u_w for some $T \in \mathcal{S}$. At least one of these paths has to be rainbow, say u_v, s_T, u_w . Then by the definition of $X(T)$ we know that bi-clique $(X(T), T \setminus X(T))$ covers edge vw in G , what concludes the proof. \square

Lemma 8. *Problem 3-SAT(φ) is reducible to $\text{BICLIQUECOVER}(G, \mathcal{S})$.*

Proof. Let φ be an instance of 3-SAT. Let v_1, \dots, v_n , resp. C_1, \dots, C_m be variables, resp. clauses of φ . Let $g: \{1, \dots, m\} \times \{1, 2, 3\} \rightarrow \{1, \dots, n\}$ be a function such that $v_{g(j,k)}$ is the variable corresponding to the k -th literal in clause C_j . If variable v has a positive, resp. negative appearance in clause C then we write $v \in C$, resp. $\neg v \in C$. We will construct graph $G = (V, E)$ and family $\mathcal{S} \subseteq 2^V$ such that φ is satisfiable if and only if (G, \mathcal{S}) is a “yes” instance of BICLIQUECOVER .

We start the construction of G by defining 2 types of vertices and 2 types of edges (see Figure 6):

$$\begin{aligned}
V_v &= \bigcup_{i=1,\dots,n} \{a_i, f_i, f_i^1, f_i^2, t_i, t_i^1, t_i^2\} \\
V_c &= \bigcup_{j=1,\dots,m} \{A_j, F_j\} \\
E_1 &= \bigcup_{i=1,\dots,n} \{a_i f_i, a_i t_i, f_i t_i^1, f_i t_i^2, t_i f_i^1, t_i f_i^2, f_i^1 t_i^1, f_i^2 t_i^2\} \\
E_2 &= \bigcup_{j=1,\dots,m} \{A_j F_j\} \cup \bigcup_{\substack{j=1,\dots,m \\ k=1,2,3}} \{A_j a_{g(j,k)}, A_j t_{g(j,k)}, F_j t_{g(j,k)}\}
\end{aligned}$$

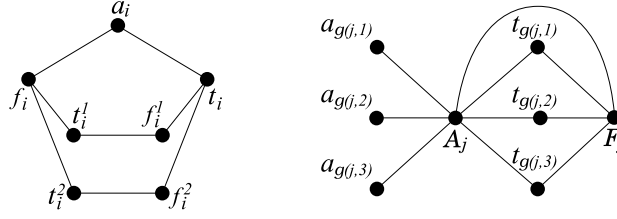


Figure 6: Edges in sets E_1 and E_2 .

Note that vertices of V_v correspond to variables while the vertices of V_c correspond to clauses of φ . We define $V = V_v \cup V_c$, $E = E_1 \cup E_2$ and $G = (V, E)$.

Next we define elements of the family $\mathcal{S} \subseteq 2^V$. For every $i = 1, \dots, n$ we define (see Figures 7 and 8):

$$\begin{aligned}
V_i^1 &= \{a_i, f_i, f_i^1, t_i, t_i^1\} \cup \bigcup_{v_i \in C_j} \{A_j\} \\
V_i^2 &= \{a_i, f_i, f_i^2, t_i, t_i^2\} \cup \bigcup_{\neg v_i \in C_j} \{A_j\}
\end{aligned}$$

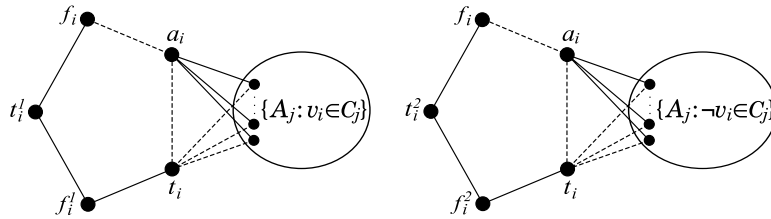


Figure 7: Sets V_i^1 and V_i^2 . By dashed lines, resp. plain lines are depicted edges, resp. uniquely coverable edges of G .

and for every $j = 1, \dots, m$ and $k = 1, 2, 3$ we define:

$$C_j^k = \{t_{g(j,k)}, A_j, F_j\}$$

We conclude the construction of \mathcal{S} by taking

$$\mathcal{S} = \bigcup_{i=1,\dots,n} \{V_i^1, V_i^2\} \cup \bigcup_{j=1,\dots,m} \{C_j^1, C_j^2, C_j^3\}$$

Note that some edges of G can be covered by only one bi-clique, since there is only one $T \in \mathcal{S}$ containing both endpoints of the given edge. We say that those edges are *uniquely coverable* (see Figures 7 and 8).

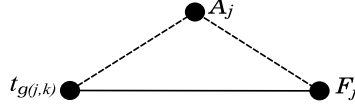


Figure 8: Set C_j^k . By dashed lines, resp. plain lines are depicted edges, resp. uniquely coverable edges of G .

At first suppose that (G, \mathcal{S}) is a “yes” instance for BICLIQUECOVER. Let $X: \mathcal{S} \rightarrow 2^V$ be the corresponding partitioning function. Define function $Y: \mathcal{S} \rightarrow 2^V$ such that $Y(T) = T \setminus X(T)$ for every $T \in \mathcal{S}$. Without loss of generality suppose that $a_i \in X(V_i^l)$ and $A_j \in X(C_j^k)$ for every feasible indices i, j, k and l . Define an evaluation $eval: \{v_1, \dots, v_n\} \rightarrow \{false, true\}$ such that $eval(v_i) = true$ if and only if $t_i \in X(V_i^1)$. We will show that $eval$ satisfies formula φ . First we prove the following four claims:

Claim (i).

Let $i \in \{1, \dots, n\}$ and $l \in \{1, 2\}$. Then one of the following holds:

- $\{f_i, f_i^l\} \subseteq X(V_i^l)$ and $\{t_i, t_i^l\} \subseteq Y(V_i^l)$
- $\{t_i, t_i^l\} \subseteq X(V_i^l)$ and $\{f_i, f_i^l\} \subseteq Y(V_i^l)$

Claim (ii).

Let $i \in \{1, \dots, n\}$, $l \in \{1, 2\}$ and let $A_j \in V_i^l$. Then $A_j \in Y(V_i^l)$.

Claim (iii).

Let $j \in \{1, \dots, m\}$ and $k \in \{1, 2, 3\}$. Then one of the following holds:

- $F_j \in X(C_j^k)$ and $t_{g(j,k)} \in Y(C_j^k)$
- $t_{g(j,k)} \in X(C_j^k)$ and $F_j \in Y(C_j^k)$

Proof of Claims (i), (ii) and (iii) follow directly from the fact that corresponding edges are uniquely coverable (see Figures 7 and 8).

Claim (iv).

Let $i \in \{1, \dots, n\}$. Then one of the following holds:

- $\{a_i, f_i\} \subseteq X(V_i^1)$, $t_i \in Y(V_i^1)$, $\{a_i, t_i\} \subseteq X(V_i^2)$ and $f_i \in Y(V_i^2)$
- $\{a_i, t_i\} \subseteq X(V_i^1)$, $f_i \in Y(V_i^1)$, $\{a_i, f_i\} \subseteq X(V_i^2)$ and $t_i \in Y(V_i^2)$

Edges $a_i f_i$ and $a_i t_i \in E(G)$ can only be covered by bi-cliques on sets V_i^1 and V_i^2 . Note that from Claim (i) follows that it is not possible to cover both edges $a_i f_i$ and $a_i t_i$ by the same bi-clique. We also know that $a_i \in X(V_i^1)$ and $a_i \in X(V_i^2)$. It means that either $\{a_i, f_i\} \subseteq X(V_i^1)$ or $\{a_i, t_i\} \subseteq X(V_i^1)$ and these two cases correspond to the two cases of our Claim (iv).

Now we will show that using $eval$ there exists at least one positively evaluated literal in every clause of formula φ . Let C_j be a clause of φ . We know that edge $A_j F_j \in E(G)$ is covered by some bi-clique. This bi-clique has to be on vertex set C_j^1 , C_j^2 or C_j^3 (since those are the only sets of \mathcal{S} containing both A_j and F_j). Suppose that $A_j \in X(C_j^k)$ and $F_j \in Y(C_j^k)$. By Claim (iii) we know that edge $A_j t_{g(j,k)} \in E(G)$ is not covered by the bi-clique on C_j^k and has to be covered by the bi-clique on V_i^1 or V_i^2 . We will show that the k -th literal of clause C_j is satisfied using the evaluation $eval$.

Let $i = g(j, k)$. If $v_i \in C_j$ (positive appearance of variable v_i) then the edge $A_j t_i \in E(G)$ has to be covered by bi-clique $(X(V_i^1), Y(V_i^1))$. That is only possible if $t_i \in X(V_i^1)$ (by Claim

(ii) we know that $A_j \in Y(V_i^1)$. It means that by the definition $eval(v_i) = true$ and k -th literal of C_j is satisfied.

If $\neg v_i \in C_j$ then the edge $A_j t_i$ has to be covered by bi-clique $(X(V_i^2), Y(V_i^2))$. That is only possible if $t_i \in X(V_i^2)$ (by Claim (ii)). From Claim (iv) we have $t_i \notin X(V_i^1)$ what means that $eval(v_i) = false$. Using the fact that v_i has a negative appearance in C_j we know that k -th literal of C_j is satisfied.

In the rest of the proof suppose that $eval: \{v_1, \dots, v_n\} \rightarrow \{false, true\}$ is a satisfying evaluation of formula φ . We will show that (G, \mathcal{S}) is a "yes" instance of BICLIQUECOVER.

For every $i \in \{1, \dots, n\}$ we define:

- if $eval(v_i) = true$: $X(V_i^1) = \{a_i, t_i, t_i^1\}$ and $X(V_i^2) = \{a_i, f_i, f_i^2\}$
- if $eval(v_i) = false$: $X(V_i^1) = \{a_i, f_i, f_i^1\}$ and $X(V_i^2) = \{a_i, t_i, t_i^2\}$

and for every $j \in \{1, \dots, m\}$ and $k \in \{1, 2, 3\}$ we define:

- if the k -th literal of C_j is evaluated to $true$: $X(C_j^k) = \{A_j, t_{g(j,k)}\}$
- if the k -th literal of C_j is evaluated to $false$: $X(C_j^k) = \{A_j, F_j\}$

Note that for every $T \in \mathcal{S}$ we define $Y(T) = T \setminus X(T)$. We will show that all edges of G are covered by some bi-clique $(X(T), Y(T))$. From the definition of $eval$ we know that edges of E_1 corresponding to variable v_i are always covered by bi-cliques on V_i^1 and V_i^2 . By definition these bi-cliques also cover all edges $A_j a_{g(j,k)}$.

Edges $t_{g(j,k)} F_j$ are covered by bi-cliques on C_j^k . To conclude the proof we need to prove that also all edges $A_j F_j$ and $A_j t_{g(j,k)}$ are covered by some bi-cliques.

From our assumption we know that every clause C_j contains at least one positively evaluated literal. For this literal we have $X(C_j^k) = \{A_j, t_{g(j,k)}\}$. It means that the edge $A_j F_j$ is covered by bi-clique $(X(C_j^k), Y(C_j^k))$.

Edge $A_j t_{g(j,k)}$ is covered by bi-clique $(X(C_j^k), Y(C_j^k))$ whenever the k -th literal of C_j is evaluated to false by $eval$. Suppose that corresponding literal is evaluated to true. Let $i = g(j, k)$. If $v_i \in C_j$, resp. $\neg v_i \in C_j$ then $eval(v_i) = true$, resp. $eval(v_i) = false$ what implies that edge $A_j t_{g(j,k)}$ is covered by bi-clique $(X(V_i^1), Y(V_i^1))$, resp. $(X(V_i^2), Y(V_i^2))$. \square

Next Theorem merges the results of Lemmata 7 and 8.

Theorem 9. *The problem RAINBOWCOLOUR($G, 2$) is NP-complete even when G is restricted to be a split graph.*

Continuing with the notations introduced in the reduction from BICLIQUECOVER(G, \mathcal{S}) to RAINBOWCOLOUR($G', 2$) (Lemma 7), consider the bipartite graph $H(A'' \dot{\cup} B', E'')$ defined as follows:

$$\begin{aligned} A'' &= \bigcup_{T \in \mathcal{S}} \{s_T\} \cup \bigcup_{e \in \bar{E}} \{x_e\} \\ B' &= \bigcup_{v \in V} \{u_v\} \\ E'' &= \bigcup_{v \in T \in \mathcal{S}} \{u_v s_T\} \cup \bigcup_{e \in \bar{E}; e=vw} \{u_v x_e, u_w x_e\} \end{aligned}$$

It is easy to see that the same proof can be modified to show that the following problem is NP-complete.

Problem 6 (BIPARTITERAINBOW(H)). Given a bipartite graph H with parts A and B , decide whether the edges of H can be 2-coloured so that there exists a rainbow path between any two vertices in part B .

The above problem is equivalent to Problem 2 (ENSUREDISTINCTROWS(C)) where n is the size of part A , m is the size of part B and C corresponds to the missing edges of H across the bipartition. To see that Problem 6 is equivalent to Problem 3 (ORTHOGONALPACKING), Fix an ordering of (a_1, \dots, a_n) of the vertices in A and associate with each vertex $v \in B$ an n -dimensional box $b(v)$ of sides (s_1, \dots, s_n) where $s_i = 1/2$ if $va_i \in E(H)$ and $s_i = 1$ otherwise. Now a $\{red, blue\}$ rainbow colouring of edges H can be interpreted as the location of the “left-bottom” corner of the boxes in a packing of them into the unit cube. In particular, a box $b(v)$ of size (s_1, \dots, s_n) occupies the space $[x_1, x_1 + s_1] \times \dots \times [x_n, x_n + s_n]$, where $x_i = 1/2$ if $va_i \in E(H)$ with colour *red* and $x_i = 0$ otherwise.